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THE EXIT PROBLEM FOR SMALL RANDOM PERTURBATIONS OF DYNAMICAL SYSTEMS WITH A HYPERBOLIC FIXED POINT

BY

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ABSTRACT

We consider the Markov diffusion process $\xi^\varepsilon(t)$ transforming when $\varepsilon = 0$ into the solution of an ordinary differential equation with a turning point $\mathcal{O}$ of the hyperbolic type. The asymptotic behavior as $\varepsilon \to 0$ of the exit time, of its expectation and of the probability distribution of exit points for the process $\xi^\varepsilon(t)$ is studied. These indicate also the asymptotic behavior of solutions of the corresponding singularly perturbed elliptic boundary value problems.

1. Introduction

In the connected bounded domain $G \subset R^n$ with the smooth boundary ∂G, let there be given a nondegenerate elliptic differential operator

$$L = \frac{1}{2} \sum_{i,j \leq n} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i \leq n} b^i(x) \frac{\partial}{\partial x_i} \tag{1.1}$$

and the first order operator

$$(B(x), \nabla) = \sum_{i \leq n} B^i(x) \frac{\partial}{\partial x_i}\ ; \tag{1.2}$$

both operators have C^3-coefficients extended smoothly into the entire space R^n so that

(i) they remain bounded functions with bounded first derivatives in R^n and

(ii) $(a^{ij}(x))$ is uniformly positive definite in R^n.

The operator $L^\varepsilon = \varepsilon^2 L + (B, \nabla)$ generates a Markov diffusion process $\xi^\varepsilon_x(t)$ being a solution of the stochastic integral equation

$$\xi^\varepsilon_x(t) = x + \int_0^t (B(\xi^\varepsilon_x(s)) + \varepsilon^2 b(\xi^\varepsilon_x(s)))ds + \varepsilon \int_0^t \sigma(\xi^\varepsilon_x(s))dw(s), \tag{1.3}$$

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where $b(x) = (b^1(x), \cdots, b^n(x))$, $\sigma(x)$ is a matrix so that $\sigma(x)\sigma^*(x) = (a^{ij}(x))$ and $w(t)$ is the n-dimensional Wiener process starting at zero (see, for instance, [2]).

The process $\xi_x^\varepsilon(t)$ is considered as a small random perturbation of the dynamical system S^t defined by the ordinary differential equation

$$\frac{d(S^t x)}{dt} = B(S^t x), \qquad S^0 x = x. \tag{1.4}$$

Let τ_x^ε be the exit time to the boundary ∂G of G for the process $\xi_x^\varepsilon(t)$, i.e.,

$$\tau_x^\varepsilon = \inf\{t : \xi_x^\varepsilon(t) \notin G\}. \tag{1.5}$$

Then the expectation $u^\varepsilon(x) = E\tau_x^\varepsilon$ satisfies the Poisson type equation (see [2])

$$L^\varepsilon u^\varepsilon = -1, \qquad u^\varepsilon \big|_{\partial G} = 0. \tag{1.6}$$

The probability distribution function $v^\varepsilon(x, t) = P\{\tau_x^\varepsilon \leqq t\}$ satisfies the parabolic equation (see [2])

$$L^\varepsilon v^\varepsilon = \partial v^\varepsilon / \partial t, \qquad v^\varepsilon \big|_{t=0} = 0, \qquad v^\varepsilon \big|_{\partial G} = 1, \tag{1.7}$$

where, as usual, $P\{\cdot\}$ denotes the probability of the event in brackets.

Let $P_x^\varepsilon(d\gamma) = P\{\xi_x^\varepsilon(\tau_x^\varepsilon) \in d\gamma\}$ be the probability distribution of the exit points of $\xi_x^\varepsilon(t)$. Then for any continuous function $\varphi(\gamma)$ on ∂G the integral

$$w_\varphi^\varepsilon(x) = \int_{\partial G} \varphi(\gamma) P_x^\varepsilon(d\gamma) \tag{1.8}$$

is the solution of the Dirichlet problem (see [2])

$$L^\varepsilon w_\varphi^\varepsilon = 0, \qquad w_\varphi^\varepsilon \big|_{\partial G} = \varphi. \tag{1.9}$$

The purpose of this paper is to study the asymptotic behavior of $P\{\tau_x^\varepsilon \leqq t\}$, $E\tau_x^\varepsilon$ and $P_x^\varepsilon(d\gamma)$ as $\varepsilon \to 0$ and therefore, as well, the asymptotics of solutions of the corresponding problems (1.6), (1.7) and (1.9).

Suppose that G contains the origin $\mathcal{O}, B(\mathcal{O}) = 0$ and $\mathcal{O}$ is the unique limit point for solutions of (1.4). Naturally, the asymptotic behavior of $u^\varepsilon, v^\varepsilon$ and w^ε as $\varepsilon \to 0$ depends on the behavior of solutions of (1.4) and, in particular, on the type of the stationary point $\mathcal{O}$.

In the case when $\mathcal{O}$ is of the center type the problem was considered in [5]. In the case of $\mathcal{O}$ being an attracting point this problem was studied in a number of papers (see, for instance, [2], [4], [8] and [10]).

This paper is concerned with the case when $\mathcal{O}$ is of the saddle type or of the repulsive type stationary point. If $x \in G$ such that $S^t x$ leaves G after some time then there is a finite

$$t(x) = \inf\{t > 0 : S^t x \in \partial G\}. \tag{1.10}$$

We shall prove that, in this case, when $\varepsilon \to 0$ then τ_x^ε tends in probability to $t(x)$, $u^\varepsilon(x)$ tends to $t(x)$ and P_x^ε tends in the weak sense to $\delta(S^{t(x)}x)$, where $\delta(z)$ is the probability measure concentrated in z.

If $x \in G$ and $S^t x \to \mathcal{O}$ as $t \to \infty$ then we shall show that there is $\lim_{\varepsilon \to 0} |\ln \varepsilon|^{-1} u^\varepsilon(x)$ independent of x. Concerning $P_x^\varepsilon(d\gamma)$ we shall see that this distribution will concentrate as $\varepsilon \to 0$ on some submanifold of the boundary ∂G.

2. Assumptions and main results

Suppose that the limit set of the dynamical system S^t in $G \cup \partial G$ contains just the one point $\mathcal{O}$, which is the origin of R^n. Assume also that for some bounded smooth vector-function $\psi(x)$,

$$B(x) = \Lambda x + \psi(x)|x|^2 \tag{2.1}$$

where Λ is a matrix having eigenvalues with nonzero real parts and at least one eigenvalue of Λ has a positive real part.

By the well known facts about stationary points (see, for instance, chapter 9 of [3]) it follows from the assumptions above that there is the decomposition

$$G \cup \partial G = \mathcal{O} \cup A_1 \cup A_2 \cup A_3, \tag{2.2}$$

where A_1 is a set of points $x \in G \cup \partial G$ such that if $x \in A_1$ then $S^u x \in G$ for $u > s$ and $S^u x \not\in G \cup \partial G$ if $u < s$ for some $s = s(x) \leqq 0$ and $S^t x \to \mathcal{O}$ as $t \to \infty$; A_2 is a set of points $x \in G \cup \partial G$ such that if $x \in A_2$ then $S^u x \in G$ for $u < s$ and $S^u x \not\in G \cup \partial G$ if $u > s$ for some $s = s(x) \geqq 0$ and $S^t x \to \mathcal{O}$ as $t \to -\infty$; A_3 is a set of points $x \in G \cup \partial G$ such that if $x \in A_3$ then $S^u x \in G$ provided $s_1 < u < s_2$ and $S^u x \not\in G \cup \partial G$ if either $u > s_2$ or $u < s_1$ for some $s_1 = s_1(x) \leqq 0$ and $s_2 = s_2(x) \geqq 0$.

REMARK 2.1. We assume that the vector field $B(x)$ is extended outside of G so that if $S^{t_1}x \in G \cup \partial G$ and $S^{t_2}x \in G \cup \partial G$ for $t_1 < t_2$ then $S^u x \in G \cup \partial G$ for all $u \in [t_1, t_2]$.

REMARK 2.2. The set A_1 can be empty. In this case A_3 is also empty.

Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of the matrix Λ and

$$\mathrm{Re}\,\lambda_1 = \cdots = \mathrm{Re}\,\lambda_\nu > \mathrm{Re}\,\lambda_{\nu+1} \geqq \cdots \geqq \mathrm{Re}\,\lambda_\mu > 0 > \mathrm{Re}\,\lambda_{\mu+1} \geqq \cdots \geqq \mathrm{Re}\,\lambda_n, \tag{2.3}$$

where $\mathrm{Re}\,a$ is the real part of a.

Denote by $\Gamma_{\max}$ the eigenspace of Λ which corresponds to the eigenvalues $\lambda_1, \cdots, \lambda_\nu$. It is known (see [3] chapter 9) that there is a ν-dimensional submanifold $W_{\max}$ tangent to $\Gamma_{\max}$ at $\mathcal{O}$ and invariant with respect to S^t. From the assumptions it follows that the intersection $Q_{\max} = W_{\max} \cap \partial G$ is not empty. If $\nu > 1$ then $Q_{\max}$ is a submanifold of $(\nu - 1)$-dimensions on the boundary ∂G. If $\nu = 1$ then $Q_{\max}$ consists of two points.

We shall prove the following results.

THEOREM 2.1. *If $x \in (\mathcal{O} \cup A_1) \setminus \partial G$ then for any $\delta > 0$,*

$$\lim_{\varepsilon \to 0} P\left\{1 - \delta < \frac{\tau_x^\varepsilon \operatorname{Re} \lambda_1}{|\ln \varepsilon|} < 1 + \delta\right\} = 1, \tag{2.4}$$

i.e., $\tau_x^\varepsilon / |\ln \varepsilon|$ tends in probability to $(\operatorname{Re} \lambda_1)^{-1}$ as $\varepsilon \to 0$. If $x \in A_2 \cup A_3$ then for any $\delta > 0$,

$$\lim_{\varepsilon \to 0} P\left\{1 - \delta < \frac{\tau_x^\varepsilon}{t(x)} < 1 + \delta\right\} = 1, \tag{2.5}$$

i.e., τ_x^ε tends in probability to $t(x)$ defined by (1.10) *and τ_x^ε is given by* (1.5).

THEOREM 2.2. *If $x \in (\mathcal{O} \cup A_1) \setminus \partial G$ then*

$$\lim_{\varepsilon \to 0} |\ln \varepsilon|^{-1} E\tau_x^\varepsilon = (\operatorname{Re} \lambda_1)^{-1}. \tag{2.6}$$

If $x \in A_2 \cup A_3$ then

$$\lim_{\varepsilon \to 0} E\tau_x^\varepsilon = t(x). \tag{2.7}$$

THEOREM 2.3. (i) *If $x \in (\mathcal{O} \cup A_1) \setminus \partial G$ then for any open subset U of ∂G such that $U \supset Q_{\max}$ one has*

$$\lim_{\varepsilon \to 0} P_x^\varepsilon(U) = 1. \tag{2.8}$$

(ii) *When $\nu = 1$, $Q_{\max}$ consists of two points z_1 and z_2 and as $\varepsilon \to 0$ then $P_x^\varepsilon(d\gamma)$ tends in the weak sense to $\frac{1}{2}(\delta(z_1) + \delta(z_2))$, where $\delta(z)$ is the probability measure concentrated at z.*

(iii) *If $x \in A_2 \cup A_3$ then P_x^ε tends in the weak sense to the probability measure $\delta(S^{t(x)}x)$ concentrated at the point $S^{t(x)}x$.*

REMARK 2.3. Theorems 2.1–2.3 by means of (1.6)–(1.9) give the asymptotic behaviour of the corresponding singularly perturbed elliptic boundary value problems. In the same way as in (4.38) below one can easily obtain also the asymptotic behaviour of the problem $L^\varepsilon u^\varepsilon = -f$, $u^\varepsilon \mid_{\partial G} = 0$ using the probabilistic representation of its solution (see [2]),

$$u^\varepsilon(x) = E\int_0^{\tau_x^\varepsilon} f(\xi_x^\varepsilon(s))ds.$$

This gives for continuous and bounded function f that

$$\lim_{\varepsilon\to 0} |\ln \varepsilon|^{-1} u^\varepsilon(x) = (\operatorname{Re}\lambda_1)^{-1} f(\mathcal{O}) \qquad \text{if } x \in (\mathcal{O}\cup A_1)\setminus \partial G \tag{2.9}$$

and

$$\lim_{\varepsilon\to 0} u^\varepsilon(x) = \int_0^{t(x)} f(S^t x)dt \qquad \text{if } x \in A_2 \cup A_3. \tag{2.10}$$

Remark 2.4. The similar results are true if the hyperbolic limit point is replaced by a hyperbolic limit circle.

3. Auxiliary Gaussian processes

For any $x \in (A_1 \cup \mathcal{O})\setminus \partial G$ define the Gaussian diffusion process $\eta_{s,z}^{\varepsilon,x}(t)$ as a solution of the following stochastic integral equation:

$$\eta_{s,z}^{\varepsilon,x}(t) = z + \int_s^t (B(S^u x) + R(S^u x)(\eta_{s,z}^{\varepsilon,x}(u) - (S^u x))du + \varepsilon\int_s^t \sigma(S^u x)dw(u), \tag{3.1}$$

where

$$R(y) = (r_{ij}(y)) = \left(\frac{\partial B^i(y)}{\partial y_j}\right). \tag{3.2}$$

Set

$$\zeta_{s,z}^{\varepsilon,x}(t) = \eta_{s,z+S^s x}^{\varepsilon,x}(t) - S^t x, \qquad \zeta_z^\varepsilon(t) = \zeta_{0,z}^{\varepsilon,\mathcal{O}}(t). \tag{3.3}$$

In what follows we use the norms

$$|z|^2 = z_1^2 + \cdots + z_n^2 \quad \text{and} \quad \|M\|^2 = \sum_{1\leq i,j\leq n} m_{ij}^2, \tag{3.4}$$

for each vector $z = (z_1, \cdots, z_n)$ and the matrix $M = (m_{ij}, 1 \leq i, j \leq n)$.

We shall need the following estimate:

Lemma 3.1. *For any $\delta > 0$ there is $C_\delta^{(0)} > 0$ such that*

$$E\,|\zeta^{\varepsilon,x}_{s,z}(t)| \leqq C^{(0)}_{\delta}(|z|+\varepsilon)e^{(\mathrm{Re}\,\lambda_1+\delta)(t-s)} \tag{3.5}$$

for each $x \in (A_1 \cup \mathcal{O})\setminus \partial G$, $z \in G$ *and* $0 \leqq s < t$.

PROOF. The process $\zeta^{\varepsilon,x}_{s,z}(t)$ satisfies the equation

$$\zeta^{\varepsilon,x}_{s,z}(t) = z + \int_s^t R(S^u x)\zeta^{\varepsilon,x}_{s,z}(u)du + \varepsilon\int_s^t \sigma(S^u x)dw(u). \tag{3.6}$$

One can see that also

$$\begin{aligned}\zeta^{\varepsilon,x}_{s,z}(t) = e^{\Lambda(t-s)}z &+ \int_s^t e^{\Lambda(t-u)}(R(S^u x)-\Lambda)\zeta^{\varepsilon,x}_{s,z}(u)du \\ &+ \varepsilon\int_s^t e^{\Lambda(t-u)}\sigma(S^u x)dw(u),\end{aligned} \tag{3.7}$$

where Λ is defined by (2.1). Indeed, taking the differentials in (3.6) and (3.7) we shall see that the solutions of equations (3.6) and (3.7) satisfy the same stochastic differential equation with the same initial condition and so coincide.

It is easy to see from (2.3) that $e^{\lambda_1 s}$ is the eigenvalue of $e^{\Lambda s}$ with the greatest absolute value. Thus by the spectral radius theorem (see, for instance, [11]) it follows that

$$\lim_{s\to\infty} \|e^{\Lambda s}\|^{1/s} = e^{\mathrm{Re}\,\lambda_1}. \tag{3.8}$$

Therefore for any δ there is $C^{(1)}_{\delta}$ such that

$$\|e^{\Lambda s}\| \leqq C^{(1)}_{\delta} e^{s(\mathrm{Re}\,\lambda_1+\delta)} \qquad \text{for all } s \geqq 0. \tag{3.9}$$

Since $x \in (A_1 \cup \mathcal{O})\setminus \partial G$ then

$$|S^u x - \mathcal{O}| \leqq C^{(2)}e^{-\alpha_0 u} \qquad \text{for all } u \geqq 0, \tag{3.10}$$

where $C^{(2)}, \alpha_0 > 0$ are independent of x and u.

By the smoothness of the coefficients of the operator L^{ε} one obtains from (3.10) that

$$\|R(S^u x)-\Lambda\| + \|\sigma(S^u x)-\sigma(\mathcal{O})\| \leqq C^{(3)}e^{-\alpha_0 u} \tag{3.11}$$

for some $C^{(3)} > 0$.

Let $M(u)$ be a random matrix-function such that for each $u \in [s,t]$ the random matrix $M(u)$ is measurable with respect to the σ-algebra F_u generated by the random values $\{w(v), 0 \leqq v \leqq u\}$; then it is well known (see [2] chapter 4.7) that

$$E\left|\int_s^t M(u)dw(u)\right|^2=\int_s^t E\|M(u)\|^2du, \tag{3.12}$$

provided the right hand side is less than infinity.

Therefore, by the Cauchy–Schwartz inequality

$$E\left|\int_s^t M(u)dw(u)\right|\leqq\left(\int_s^t E\|M(u)\|^2du\right)^{1/2}. \tag{3.13}$$

Employing (3.13) with $M(u)=e^{\Lambda(t-u)}\sigma(S^u x)$ together with (3.9) and (3.11) we obtain from (3.7) that

$$\begin{aligned}&E\,|\zeta^{\varepsilon,x}_{s,z}(t)|\\ &\leqq e^{(t-s)(\mathrm{Re}\,\lambda_1+\delta)}\left(C^{(1)}_\delta|z|+C^{(1)}_\delta C^{(3)}\int_s^t e^{-\alpha_0 u}e^{-(u-s)(\mathrm{Re}\,\lambda_1+\delta)}E\,|\zeta^{\varepsilon,x}_{s,z}(u)|\,du+\varepsilon C^{(4)}\right),\end{aligned} \tag{3.14}$$

for some $C^{(4)}>0$.

Therefore by Gronwall's inequality (see [3]) for the function

$$a_1(t)=e^{-(\mathrm{Re}\,\lambda_1+\delta)(t-s)}E\,|\zeta^{\varepsilon,x}_{s,z}(t)| \tag{3.15}$$

one gets (3.5) with $C^{(0)}_\delta=(C^{(1)}_\delta+C^{(4)})\exp(C^{(1)}_\delta C^{(3)}/\alpha_0)$.

The next estimate we shall need is the following result:

LEMMA 3.2. *For any $\delta>0$ there is $C^{(5)}_\delta$ so that*

$$E\,|\zeta^{\varepsilon,x}_{s,z}(t)-\zeta^{\varepsilon,\mathcal{O}}_{s,z}(t)|\leqq C^{(5)}_\delta(|z|+\varepsilon)e^{-\alpha_0 s}e^{(\mathrm{Re}\,\lambda_1+\delta)(t-s)}, \tag{3.16}$$

for each $x\in(A_1\cup\mathcal{O})\setminus\partial G$, $z\in G$ and $0\leqq s<t$.

PROOF. From (3.5), (3.7), (3.9), (3.11), and (3.13) we easily find

$$\begin{aligned}E\,|\zeta^{\varepsilon,x}_{s,z}(t)-\zeta^{\varepsilon,\mathcal{O}}_{s,z}(t)|&\leqq C^{(1)}_\delta C^{(3)}\int_s^t e^{(\mathrm{Re}\,\lambda_1+\delta)(t-u)}e^{-\alpha_0 u}E\,|\zeta^{\varepsilon,x}_{s,z}(u)|\,du\\ &\quad+C^{(1)}_\delta C^{(6)}\varepsilon e^{(\mathrm{Re}\,\lambda_1+\delta)(t-s)}e^{-\alpha_0 s}\\ &\leqq C^{(5)}_\delta(|z|+\varepsilon)e^{-\alpha_0 s}e^{(\mathrm{Re}\,\lambda_1+\delta)(t-s)}\end{aligned} \tag{3.17}$$

with some $C^{(6)}>0$ and $C^{(5)}_\delta=C^{(0)}_\delta C^{(1)}_\delta C^{(3)}\alpha_0^{-1}+C^{(1)}_\delta C^{(6)}$ that proves (3.16).

Now we need the following estimates for the process $\zeta^\varepsilon_z(t)$ defined by (3.3):

LEMMA 3.3. (i) *There is $K_0>0$ such that*

$$P\{|\zeta^\varepsilon_z(t)|\leqq r\}\leqq K_0\varepsilon^{-\nu}r^n e^{-t\nu\,\mathrm{Re}\,\lambda_1}, \tag{3.18}$$

for any $z\in G$ and $\varepsilon,r,t>0$.

(ii) *There is* $K_1 > 0$ *and* $\delta_0 > 0$ *such that*

$$E\,\mathrm{dist}(\zeta_z^\varepsilon(t), \Gamma_{\max}) \leqq K_1(\varepsilon + \mathrm{dist}(z, \Gamma_{\max}))e^{(\mathrm{Re}\,\lambda_1 - \delta_0)t}, \tag{3.19}$$

for every $z \in G$ *and* $\varepsilon, t > 0$, *where, recall,* $\Gamma_{\max}$ *is the eigenspace of* Λ *corresponding to the eigenvalues* $\lambda_1, \cdots, \lambda_\nu$.

PROOF. Let Γ_1 be the eigenspace of the matrix Λ corresponding to the eigenvalues having real parts less than $\mathrm{Re}\,\lambda_1$. One can choose vectors $\xi^{(\nu+1)}, \cdots. \xi^{(n)}$ in the linear space Γ_1 so that

$$\langle \xi^{(i)}, \xi^{(j)} \rangle = (A^{-1}\xi^{(i)}, \xi^{(j)}) = \delta_{ij}, \tag{3.20}$$

where $A = (a^{ij}(\mathcal{O}))$ and $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$.

Consider now a basis $\xi^{(1)}, \cdots, \xi^{(n)}$ of the whole R^n such that $\xi^{(\nu+1)}, \cdots, \xi^{(n)}$ are the same as above and (3.20) holds for all $0 \leqq i, j \leqq n$. For any vector $\xi \in R^n$ there is a unique representation $\xi = \hat{\xi} + \hat{\hat{\xi}}$ with $\hat{\hat{\xi}} \in \Gamma_1$ and $\hat{\xi} \in \Gamma_0$, where Γ_0 is spanned by $\xi^{(1)}, \cdots, \xi^{(\nu)}$.

Since Γ_1 is an eigenspace of Λ then $\Lambda\xi^{(i)} \in \Gamma_1$ for all $i = \nu + 1, \cdots, n$. Hence, equation (3.6) can be written for $\zeta_z^\varepsilon(t)$ in the new basis in the form

$$\hat{\xi}_z^\varepsilon(t) = \hat{z} + \int_0^t \Lambda_0 \hat{\zeta}_z^\varepsilon(u)du + \varepsilon\hat{w}(t), \tag{3.21}$$

$$\hat{\hat{\zeta}}_z^\varepsilon(t) = \hat{\hat{z}} + \int_0^t (\Lambda_2\hat{\zeta}_z^\varepsilon(u) + \Lambda_1\hat{\hat{\zeta}}_z^\varepsilon(u))du + \varepsilon\hat{\hat{w}}, \tag{3.22}$$

where Λ_0 is a $(\nu \times \nu)$-matrix, Λ_2 is a $((n-\nu)\times\nu)$-matrix, Λ_1 is a $((n-\nu)\times(n-\nu))$-matrix and $\hat{w}(t)$ and $\hat{\hat{w}}(t)$ are the Wiener processes (with respect to the inner product (3.20)) in Γ_0 and Γ_1, respectively.

It is clear that

$$P\{|\zeta_z^\varepsilon(t)| \leqq r\} \leqq P\{|\hat{\zeta}_z^\varepsilon(t)| \leqq K_2 r\} = \int_{\{y:|y|\leqq K_2 r\}} \hat{q}^\varepsilon(t, z, y)dy, \tag{3.23}$$

where $\hat{q}^\varepsilon(t, z, y)$ is the transition density of the process $\hat{\zeta}_z^\varepsilon(t)$ and $K_2 > 0$ is a constant depending just on the matrix A. This transition density has the explicit form

$$\begin{aligned} \hat{q}^\varepsilon(t, z, y) &= (2\pi\varepsilon^2)^{-\nu/2}\left(\det \int_0^t e^{(t-s)\Lambda_0}e^{(t-s)\Lambda_0^*}ds\right)^{-1/2} \\ &\times \exp\left\{-\frac{1}{2}\varepsilon^{-2}\left\langle\left(\int_0^t e^{(t-s)\Lambda_0}e^{(t-s)\Lambda_0^*}ds\right)^{-1}(y - e^{t\Lambda_0}z), (y - e^{t\Lambda_0}z)\right\rangle\right\}. \end{aligned} \tag{3.24}$$

Obviously, the eigenvalues of Λ_0 are $\lambda_1, \cdots, \lambda_\nu$ and so by inequality (5.20) of [6],

$$K_3^{-1} e^{2t\nu \operatorname{Re}\lambda_1} \leqq \det \left(\int_0^t e^{(t-s)\Lambda_0} e^{(t-s)\Lambda_0^*} ds \right) \leqq K_3 e^{2t\nu \operatorname{Re}\lambda_1}, \tag{3.25}$$

for some $K_3 > 0$. Thus

$$\hat{q}^\varepsilon(t, z, y) \leqq (2\pi)^{-\nu/2} \varepsilon^{-\nu} K_3^{1/2} e^{-t\nu \operatorname{Re}\lambda_1} \tag{3.26}$$

which, together with (3.23), gives (3.18).

To prove item (ii) pick a basis $\psi^{(1)}, \cdots, \psi^{(n)}$ of R^n satisfying (3.20) and such that $\psi^{(1)}, \cdots, \psi^{(\nu)} \in \Gamma_{\max}$. For any vector $\psi \in R^n$ one has the unique representation $\psi = \tilde{\psi} + \tilde{\tilde{\psi}}$ with $\tilde{\psi} \in \Gamma_{\max}$ and $\tilde{\tilde{\psi}} \in \Gamma_2$, where Γ_2 is spanned by $\psi^{(\nu+1)}, \cdots, \psi^{(n)}$.

It is clear that $\Lambda\psi^{(i)} \in \Gamma_{\max}$ for all $i = 1, \cdots, \nu$. Thus equation (3.5) for $\zeta_z^\varepsilon(t)$ in this basis can be written as follows:

$$\tilde{\zeta}_z^\varepsilon(t) = \tilde{z} + \int_0^t (\Lambda_{\max} \tilde{\zeta}_z^\varepsilon(u) + \Lambda_3 \tilde{\tilde{\zeta}}_z^\varepsilon(u)) du + \varepsilon \tilde{w}(t), \tag{3.27}$$

$$\tilde{\tilde{\zeta}}_z^\varepsilon(t) = \tilde{\tilde{z}} + \int_0^t \Lambda_4 \tilde{\tilde{\zeta}}_z(u) du + \varepsilon \tilde{\tilde{w}}(t), \tag{3.28}$$

where $\Lambda_{\max}$ is a $(\nu \times \nu)$-matrix, Λ_3 is a $(\nu \times (n-\nu))$-matrix, Λ_4 is a $((n-\nu) \times (n-\nu))$-matrix and $\tilde{w}(t)$ and $\tilde{\tilde{w}}(t)$ are the Wiener processes in $\Gamma_{\max}$ and Γ_2, respectively, which correspond to the inner product $\langle \ , \ \rangle$.

Obviously,

$$K_4^{-1} |\tilde{\tilde{\zeta}}_z^\varepsilon(t)| \leqq \operatorname{dist}(\zeta_z^\varepsilon(t), \Gamma_{\max}) \leqq K_4 |\tilde{\tilde{\zeta}}_z^\varepsilon(t)|, \tag{3.29}$$

for some $K_4 > 0$ depending just on the matrix A.

It is clear that the eigenvalues of Λ_4 are $\lambda_{\nu+1}, \cdots, \lambda_n$ and so for any $\delta > 0$ there is $C_\delta^{(7)}$ such that

$$\| e^{t\Lambda_4} \| \leqq C_\delta^{(7)} e^{t(\operatorname{Re}\lambda_{\nu+1} + \delta)}, \tag{3.30}$$

for any $t \geqq 0$.

The solution of (3.28) has the form

$$\tilde{\tilde{\zeta}}_z^\varepsilon(t) = e^{t\Lambda_4} \tilde{\tilde{z}} + \varepsilon \int_0^t e^{\Lambda_4(t-u)} d\tilde{\tilde{w}}(u). \tag{3.31}$$

Taking into account (2.3), (3.13) and (3.30) we easily obtain

$$E |\tilde{\tilde{\zeta}}_z^\varepsilon(t)| \leqq C_{\delta_0}^{(7)} (\varepsilon + |\tilde{\tilde{z}}|) e^{(\operatorname{Re}\lambda_1 - \delta_0)t} (1 + (\operatorname{Re}\lambda_1)^{-1/2}) \tag{3.32}$$

with

$$\delta_0 = \min\left(\frac{\operatorname{Re}\lambda_1 - \operatorname{Re}\lambda_{\nu+1}}{2}, \frac{\operatorname{Re}\lambda_1}{2}\right). \tag{3.33}$$

Now from (3.29) and (3.32), (3.19) follows with $K_1 = K_4 C^{(7)}_{\delta_0}(1+K_4)(1+(\operatorname{Re}\lambda_1)^{-1/2})$.

Set

$$t_\gamma(\varepsilon) = (\operatorname{Re}\lambda_1 + \gamma)^{-1}|\ln\varepsilon| \quad \text{and} \quad r_\gamma(\varepsilon) = \varepsilon^{\gamma(\operatorname{Re}\lambda_1+\gamma)^{-1}} \quad \text{for } \gamma > -\operatorname{Re}\lambda_1. \tag{3.34}$$

Actually, we shall need just the following result:

Lemma 3.4. *There are some positive constants $C^{(8)}$, $\gamma_0 < 1$, γ_1, γ_2 and γ_3 such that for any $\delta > 0$ small enough and $x \in (A_1 \cup \mathcal{O})\setminus\partial G$ there is $C^{(9)}_\delta$ so that*

$$P\{|\eta^{\varepsilon,x}_{0,x}(t_{\gamma_0}(\varepsilon))| \geqq \varepsilon^{-\delta} r_{\gamma_0}(\varepsilon)\} \leqq C^{(9)}_\delta \varepsilon^{\gamma_2\delta}, \tag{3.35}$$

$$P\{|\eta^{\varepsilon,x}_{0,x}(t_{\gamma_0}(\varepsilon))| \leqq \varepsilon^{\delta} r_{\gamma_0}(\varepsilon)\} \leqq C^{(9)}_\delta \varepsilon^{\gamma_2\delta}, \tag{3.36}$$

$$P\{\operatorname{dist}(\eta^{\varepsilon,x}_{0,x}(t_{\gamma_0}(\varepsilon)), \Gamma_{\max}) \geqq \varepsilon^{\gamma_1} r_{\gamma_0}(\varepsilon)\} \leqq C^{(8)} \varepsilon^{\gamma_3}, \tag{3.37}$$

where we can take, for instance, $\gamma_0 = \frac{1}{4}\min(\alpha_0, 1)$ with α_0 defined in (3.10) *and the other constants will be found below.*

Proof. By (3.3) and (3.10) one has

$$|\zeta^{\varepsilon,x}_{0,\mathcal{O}}(t)| \geqq |\eta^{\varepsilon,x}_{0,x}(t)| - |S^t x| \geqq |\eta^{\varepsilon,x}_{0,x}(t)| - C^{(2)} e^{-\alpha_0 t}. \tag{3.38}$$

Setting here $t = t_{\gamma_0}(\varepsilon)$ we shall obtain (3.35) from (3.38), Chebyshev's inequality and (3.5) used with δ replaced by $\frac{1}{2}\delta(\operatorname{Re}\lambda_1 + \gamma_0)$.

For $x = \mathcal{O}$ relation (3.36) is given by (3.18). We shall prove here (3.36) for $x \in A_1 \setminus \partial G$ satisfying (3.10) with $\alpha_0 > \operatorname{Re}\lambda_1$. One can verify (3.36) for other $x \in A_1$ in the same way as in §4 of [7] by considering the explicit form of the transition density of the Gaussian process $\zeta^{\varepsilon,x}_{0,\mathcal{O}}(t)$ (see Appendix).

By (3.3), (3.5), (3.10), (3.16), (3.18), (3.38), the Markov property and Chebyshev's inequality one can see that

$$\begin{aligned} P\{|\eta^{\varepsilon,x}_{0,x}(t)| \leqq r\} &\leqq P\{|\zeta^{\varepsilon,x}_{0,\mathcal{O}}(t)| \leqq r + C^{(2)} e^{-\alpha_0 t}\} \\ &= EP\{|\zeta^{\varepsilon,x}_{s,\zeta^{\varepsilon,x}_{0,\mathcal{O}}(s)}(t)| \leqq r + C^{(2)} e^{-\alpha_0 t}\} \\ &\leqq EP\{|\zeta^{\varepsilon}_{\zeta^{\varepsilon,x}_{0,\mathcal{O}}(s)}(t-s)| \leqq 2(r + C^{(2)} e^{-\alpha_0 t})\} \\ &\quad + EP\{|\zeta^{\varepsilon,x}_{s,\zeta^{\varepsilon,x}_{0,\mathcal{O}}(s)}(t) - \zeta^{\varepsilon,\mathcal{O}}_{s,\zeta^{\varepsilon,x}_{0,\mathcal{O}}(s)}(t)| \geqq r + C^{(2)} e^{-\alpha_0 t}\} \\ &\leqq K_0 \varepsilon^{-\nu} 2^\nu (r + C^{(2)} e^{-\alpha_0 t})^\nu e^{-(t-s)\nu \operatorname{Re}\lambda_1} \\ &\quad + 2C^{(5)}_{\delta_1}(C^{(0)}_{\delta_1} + 1)\cdot\varepsilon\cdot(r + C^{(2)} e^{-\alpha_0 t})^{-1}\cdot\exp[(\operatorname{Re}\lambda_1 + \delta_1)t - \alpha_0 s]. \end{aligned} \tag{3.39}$$

Given δ take here $t_{\gamma_0}=t_{\gamma_0}(\varepsilon)$, $r=\varepsilon^{\delta}r_{\gamma_0}(\varepsilon)$ and, for instance, $s=\frac{1}{2}\delta\,|\ln\varepsilon\,|(\alpha_0^{-1}+(\operatorname{Re}\lambda_1)^{-1})$, $\delta_1=\frac{1}{8}\delta(\alpha_0-\operatorname{Re}\lambda_1)(\operatorname{Re}\lambda_1+\gamma_0)(\operatorname{Re}\lambda_1)^{-1}$, then (3.36) follows from (3.39), provided $\alpha_0>\operatorname{Re}\lambda_1$.

Since $\Gamma_{\max}$ is a hyperplane we obtain by (3.3), (3.5), (3.10), (3.16), (3.19), (3.38), the Markov property and Chebyshev's inequality that

$$
\begin{aligned}
&P\{\operatorname{dist}(\eta_{0,x}^{\varepsilon,x}(t),\Gamma_{\max})>r\}\\
&\leqq P\{\operatorname{dist}(\zeta_{0,\mathcal{O}}^{\varepsilon,x}(t),\Gamma_{\max})>r-C^{(2)}e^{-\alpha_0 t}\}\\
&\leqq EP\{\operatorname{dist}(\zeta^{\varepsilon}_{\zeta_{0,\mathcal{O}}^{\varepsilon,x}(s)}(t-s),\Gamma_{\max})>\tfrac{1}{2}(r-C^{(2)}e^{-\alpha_0 t})\}\\
&\quad+EP\{|\,\zeta^{\varepsilon,x}_{s,\zeta_{0,\mathcal{O}}^{\varepsilon,x}(s)}(t)-\zeta^{\varepsilon,\mathcal{O}}_{s,\zeta_{0,\mathcal{O}}^{\varepsilon,x}(s)}(t)|>\tfrac{1}{2}(r-C^{(2)}e^{-\alpha_0 t})\}\\
&\leqq 2K_1(C_{\delta_2}^{(0)}+1)\varepsilon(r-C^{(2)}e^{-\alpha_0 t})^{-1}\exp[\delta_2 s+t\operatorname{Re}\lambda_1-\delta_0(t-s)]\\
&\quad+2C_{\delta_2}^{(5)}(C_{\delta_2}^{(0)}+1)\varepsilon(r-C^{(2)}e^{-\alpha_0 t})^{-1}\exp[t(\operatorname{Re}\lambda_1+\delta_2)-\alpha_0 s].
\end{aligned}
\tag{3.40}
$$

Put here $t=t_{\gamma_0}(\varepsilon)$, $\gamma_1=\frac{1}{4}\min(\delta_0,\alpha_0)(\operatorname{Re}\lambda_1+\gamma_0)^{-1}$, $r=\varepsilon^{\gamma_1}r_{\gamma_0}(\varepsilon)$, $s=t/2$ and $\delta_2=\frac{1}{8}\min(\delta_0,\alpha_0)$, then (3.37) follows from (3.40).

4. Proof of Theorems 2.1–2.3

We start this section with the following result proved in [6] (Lemma 4.1 and Corollary 4.1).

LEMMA 4.1. *There exist $d_0, r_0>0$ which depend just on G and the flow S^t such that, for any sequence of points $z_0,\cdots,z_m\in G$ satisfying the property:*

$$
\begin{gathered}
S^1z_i\in G \quad \textit{and} \quad z_{i+1}\in U_r(S^1z_i)\\
\textit{for all } i=0,\cdots,m-1 \textit{ with some } r\leqq r_0,
\end{gathered}
\tag{4.1}
$$

there is a point $y\in G$ such that

$$
z_i\in U_{d_0r}(S^iy),\qquad i=0,\cdots,m,
\tag{4.2}
$$

where $U_r(z)=\{v:|z-v|\leqq r\}$.

Another result about the dynamical system S^t that we shall need here is the following:

LEMMA 4.2. *There is an open domain $\tilde G\supset G\cup\partial G$ such that for any $\delta>0$ there is $C_\delta^{(10)}>0$ with the property: if for some points x and y,*

$$
S^ux\in\tilde G \quad \textit{and} \quad S^uy\in\tilde G \qquad \textit{for all } u\in[0,t]
\tag{4.3}
$$

then

$$\sup_{0 \leq u \leq t} |S^u x - S^u y| \leq C_\delta^{(10)} e^{(\mathrm{Re}\,\lambda_1 + \delta)t} |x - y|. \tag{4.4}$$

PROOF. Choose some domain $\tilde{G} \supset G \cup \partial G$ such that the decomposition (2.2) holds for $\tilde{G}$, as well. It is possible since $B(z)$ was extended smoothly into the whole of R^n.

One can write

$$S^u z = e^{u\Lambda} z + \int_0^u e^{(u-v)\Lambda}(B(S^w z) - \Lambda S^v z)dv. \tag{4.5}$$

Indeed, differentiating both sides in (4.5) we shall get that the solution of (4.5) satisfies (1.4).

Substituting x and y into (4.5) we deduce from (3.9) that

$$\begin{aligned} |S^u x - S^u y| &\leq C_\delta^{(1)} e^{(\mathrm{Re}\,\lambda_1 + \delta)u} |x - y| \\ &+ C_\delta^{(1)} \int_0^u e^{(\mathrm{Re}\,\lambda_1 + \delta)(u-v)} |B(S^v x) - \Lambda S^v x - B(S^v y) + \Lambda S^v y| dv. \end{aligned} \tag{4.6}$$

By lemma 4.3 of [6] it follows from (4.3) that

$$|S^v x| + |S^v y| \leq K_5 \max(e^{-\alpha_1 v}, e^{-\alpha_1 (t-v)}) \tag{4.7}$$

for some $K_5, \alpha_1 > 0$ independent of $x, y \in G$. Using (2.1), (4.6), (4.7) and the smoothness of ψ in (2.1) we easily obtain that for some $\tilde{K}_5 > 0$,

$$\begin{aligned} \sup_{0 \leq u \leq t} |S^u x - S^u y| &\leq C_\delta^{(1)} e^{(\mathrm{Re}\,\lambda_1 + \delta)t} |x - y| \\ &+ C_\delta^{(1)} \tilde{K}_5 e^{(\mathrm{Re}\,\lambda_1 + \delta)t} \int_0^t \max(e^{-\alpha_1 u}, e^{-\alpha_1 (t-u)}) e^{-(\mathrm{Re}\,\lambda_1 + \delta)u} \sup_{0 \leq v \leq u} |S^v x - S^v y| du. \end{aligned} \tag{4.8}$$

Applying Gronwall's inequality (see [3]) to the function

$$a_2(t) = e^{-(\mathrm{Re}\,\lambda_1 + \delta)t} \sup_{0 \leq u \leq t} |S^u x - S^u y|$$

considered in (4.8), we deduce (4.4).

We shall need also the following important result:

LEMMA 4.3. *There are positive constants K_6, α_2 and α_3 such that for any $\beta > 0$ one can find $\varepsilon_0(\beta) > 0$ and $K_7(\beta) > 0$ so that the following is true*:

$$\begin{aligned} Q_1^\varepsilon(x, T) &\equiv P^\varepsilon \left\{ \inf_{z \in U_{\varepsilon^{1-\delta}}(x)} \sup_{0 \leq t \leq \min(T, \tau_x^\varepsilon)} |\xi_x^\varepsilon(t) - S^t z| > \varepsilon^{1-\delta} \right\} \\ &\leq K_6 (\min(T, \varepsilon^{-\beta}))^2 \exp(-\alpha_2/\varepsilon^{2\delta}) + K_7(\beta) \exp(-\alpha_3/\varepsilon^\beta), \end{aligned} \tag{4.9}$$

for any $x \in G$ and positive $\delta < 1$ and T, provided $0 < \varepsilon \leqq \varepsilon_0(\beta)$, where, recall, τ_x^ε is the exit time to the boundary ∂G for the process $\xi_x^\varepsilon(t)$.

PROOF. (Cf. section 5 of [6].) In the author's paper [6] it was actually proved that there is a positive constant $\alpha_4 > 0$ such that for any $\beta > 0$ one can find $\varepsilon_1(\beta) > 0$ and $K_8(\beta) > 0$ so that

$$P\{\tau_x^\varepsilon > t\} \leqq K_8(\beta)\exp(-\alpha_4 t) \quad \text{for all } t \geqq \varepsilon^{-\beta}, \quad \text{provided } \varepsilon \leqq \varepsilon_1(\beta). \tag{4.10}$$

Therefore, if $\varepsilon \leqq \varepsilon_1(\beta)$ then

$$0 \leqq Q_1^\varepsilon(x, T) - Q_1^\varepsilon(x, \min(T, \varepsilon^{-\beta})) \leqq K_8(\beta)\exp(-\alpha_4 \varepsilon^{-\beta}). \tag{4.11}$$

Set $N =$ integral part of $\min(T, \varepsilon^{-\beta})$ and consider

$$Q_2^\varepsilon(x, N+1) \equiv P\left\{ \inf_{z \in U_{\varepsilon^{1-\delta}}(x)} \sup_{\substack{k\text{ — integer} \\ 0 \leq k \leq \min(\tau_x^\varepsilon, N+1)}} |\xi_x^\varepsilon(k) - S^k z| > 2d_0 d_1 r(\varepsilon) \right\},$$

where d_0 is defined in Lemma 4.1,

$$r(\varepsilon) = \varepsilon^{1-\delta}/4d_0 d_1^2, \qquad d_1 = \sup_{\substack{z \in G \cup \partial G \\ -1 \leq u \leq 1}} \|DS^u(z)\| \tag{4.12}$$

and $DS^u(z)$ is the Jacobian matrix of S^u at z.

One can easily obtain from [1] (see formulas (5.7) and (5.8) of [7]) that

$$\begin{aligned} &Q_1^\varepsilon(x, \min(T, \varepsilon^{-\beta})) - Q_2^\varepsilon(x, N+1) \\ &\leqq \min(T, \varepsilon^{-\beta}) \sup_{\substack{k\text{ — integer} \\ 0 \leq k \leq N, z \in G}} P\Big\{ |\xi_x^\varepsilon(k) - z| \leqq \varepsilon^{1-\delta}/2d_1, \\ &\qquad |\xi_x^\varepsilon(k+1) - S^1 z| \leqq \varepsilon^{1-\delta}/2d_1 \text{ and } \sup_{0 \leq t \leq 1} |\xi_x^\varepsilon(k+t) - S^t z| > \varepsilon^{1-\delta} \Big\} \\ &\leqq \min(T, \varepsilon^{-\beta}) K_9 \exp(-\alpha_5/\varepsilon^{2\delta}), \end{aligned} \tag{4.13}$$

for some $K_9, \alpha_5 > 0$.

Next we have

$$\begin{aligned} P\{\tau_x^\varepsilon \geqq m\} &= \int_G \cdots \int_G p^\varepsilon(1, x, z_1) \cdots p^\varepsilon(1, z_{m-1}, z_m) dz_1 \cdots dz_m \\ &= I^\varepsilon(x, m) + R^\varepsilon(x, m), \end{aligned} \tag{4.14}$$

where $p^\varepsilon(t, x, y)$ is the transition density of the process $\xi_x^\varepsilon(t)$ with the absorption on the boundary ∂G, i.e., in fact, of the process $\tilde{\xi}_x^\varepsilon(t) = \xi_x^\varepsilon(\min(t, \tau_x^\varepsilon))$, and

(4.15) $$I^\varepsilon(x,m)=\int_{U_{r(\varepsilon)}(S^1x)}\cdots\int_{U_{r(\varepsilon)}(S^1z_{m-1})}p^\varepsilon(1,x,z_1)\cdots p^\varepsilon(1,z_{m-1},z_m)dz_1\cdots dz_m.$$

From estimates of [1] it follows that

(4.16) $$R^\varepsilon(x,m)\leqq m\sup_{z\notin U_{r(\varepsilon)}(S^1y)}p^\varepsilon(1,y,z)\leqq K_{10}m\exp(-\alpha_6/\varepsilon^{2\delta}),$$

for some $K_{10},\alpha_6>0$.

Notice that the integration in the integral $I^\varepsilon(x,m)$ in (4.15) is over all sequences $\omega=(z_0,z_1,\cdots,z_m)$, $z_0=x$, satisfying (4.1) with $r=r(\varepsilon)$. Therefore by Lemma 4.1 for any such sequence ω there is a point y^ω so that

(4.17) $$\sup_{\substack{k\text{ — integer}\\0\leqq k\leqq m}}|z_k-S^ky^\omega|\leqq d_0r(\varepsilon),$$

where $\omega=(z_0,\cdots,z_m)$, $z_0=x$ and $z_k\in G$ for all $k=0,\cdots,m$.

Denote

$$Q_3^\varepsilon(x;r;l;m)=P\left\{\tau_x^\varepsilon\geqq m\text{ and }\inf_{z\in U_{\varepsilon^{1-\delta}}(x)}\sup_{\substack{k\text{ — integer}\\0\leqq k\leqq l}}|\xi_x^\varepsilon(k)-S^kz|>r\right\}.$$

Then by (4.14)–(4.17) we get

(4.18) $$Q_3^\varepsilon(x;d_0r(\varepsilon);m,m)\leqq K_{10}m\exp(-\alpha_6/\varepsilon^{2\delta}).$$

Recall that the coefficients of the stochastic equation (1.3) are extended into the entire space R^n as bounded and smooth functions and so we can consider $\xi_x^\varepsilon(t)$ after the moment τ_x^ε, as well, and S^t can be applied to any $z\in R^n$.

As in (4.16) we obtain from [11] and the definition of d_1 in (4.12) that

(4.19) $$\begin{aligned}&\sup_{z\in G\cup\partial G}P\{|\xi_x^\varepsilon(k)-z|\leqq d_0r(\varepsilon)\text{ and }|\xi_x^\varepsilon(k+1)-S^1z|>2d_0d_1r(\varepsilon)\}\\&\leqq K_{11}\exp(-\alpha_7/\varepsilon^{2\delta}),\end{aligned}$$

for some $K_{11},\alpha_7>0$.

Now (4.18) together with (4.19) gives

(4.20) $$Q^\varepsilon(x,2d_0d_1r(\varepsilon),m,m+1)\leqq K_{12}m\exp(-\alpha_8/\varepsilon^{2\delta}),$$

for some $K_{12},\alpha_8>0$.

Obviously

$$Q_2^\varepsilon(x,N+1)\leqq\sum_{m=0}^{N+1}Q_3^\varepsilon(x,2d_0d_1r(\varepsilon),m,m+1)$$

and so by (4.11), (4.13) and (4.20) we get (4.9) which proves Lemma 4.3.

From Lemmas 4.2 and 4.3 we obtain immediately the following statement:

COROLLARY 4.1. *For any positive* $\delta, \gamma < 1$ *there is* $C^{(11)}_{\delta,\gamma} > 0$ *such that*

$$E \sup_{0 \leq u \leq \min(t, \tau^\varepsilon_x)} |\xi^\varepsilon_x(u) - S^u x|^2 \leqq C^{(11)}_{\delta,\gamma} \varepsilon^{2-\gamma} \exp[2t(\operatorname{Re} \lambda_1 + \delta)] \tag{4.21}$$

for every $x \in (\mathcal{O} \cup A_1) \setminus \partial G$ *and* $t > 0$, *provided* ε *is small enough.*

Now we are able to compare the processes $\xi^\varepsilon_x(t)$ and $\eta^{\varepsilon,x}_{0,x}(t)$.

LEMMA 4.4. *For any positive* $\delta, \gamma < 1$ *there is* $C^{(12)}_{\delta,\gamma} > 0$ *such that*

$$E |\xi^\varepsilon_x(t) - \eta^{\varepsilon,x}_{0,x}(t)| \chi_{t \leq \tau^\varepsilon_x} \leqq C^{(12)}_{\delta,\gamma} \varepsilon^{2-\gamma} \exp[2t(\operatorname{Re} \lambda_1 + \delta)] \tag{4.22}$$

provided $x \in (\mathcal{O} \cup A_1) \setminus \partial G$, $t > 0$ *and* ε *is small enough.*

PROOF. One can verify directly that the process $\xi^\varepsilon_x(t)$ satisfying (1.3) is also the solution of the following equation:

$$\begin{aligned} \xi^\varepsilon_x(t) - S^t x &= \int_0^t e^{\Lambda(t-u)}(B(\xi^\varepsilon_x(u)) - B(S^u x) - \Lambda(\xi^\varepsilon_x(u) - S^u x))du \\ &\quad + \varepsilon^2 \int_0^t e^{\Lambda(t-u)} b(\xi^\varepsilon_x(u))du + \varepsilon \int_0^t e^{(t-u)\Lambda} \sigma(\xi^\varepsilon_x(u))dw(u). \end{aligned} \tag{4.23}$$

By the C^2-smoothness of coefficients $B(z)$ we can write

$$B(z) = B(S^u x) + R(S^u x)(z - S^u x) + \tilde{\psi}(z, x, u)|z - S^u x|^2, \tag{4.24}$$

where $\tilde{\psi}$ is a bounded function when $x \in \mathcal{O} \cup A_1$ and $z \in G \cup \partial G$.

Employing (3.3), (3.7), (3.9), (3.11) and (4.21), (4.23), (4.24) we obtain

$$\begin{aligned} &E |\xi^\varepsilon_x(t) - \eta^{\varepsilon,x}_{0,x}(t)| \chi_{t \leq \tau^\varepsilon_x} \\ &\leqq C^{(1)}_\delta C^{(3)} \int_0^t e^{(\operatorname{Re}\lambda_1 + \delta)(t-u)} e^{-\alpha_0 u} E |\xi^\varepsilon_x(u) - \eta^{\varepsilon,x}_{0,x}(u)| \chi_{u \leq \tau^\varepsilon_x} du \\ &\quad + \tilde{K}_{12} C^{(1)}_\delta C^{(11)}_{\delta,\gamma} \varepsilon^{2-\gamma} \int_0^t e^{(\operatorname{Re}\lambda_1 + \delta)(t+u)} du \\ &\quad + \tilde{K}_{12} C^{(1)}_\delta \varepsilon^2 \int_0^t e^{(\operatorname{Re}\lambda_1 + \delta)(t-u)} du + J^\varepsilon(x, t), \end{aligned} \tag{4.25}$$

for some $\tilde{K}_{12} > 0$, where

$$J^\varepsilon(x, t) \equiv \varepsilon E \chi_{t \leq \tau^\varepsilon_x} \left| \int_0^t e^{(t-u)\Lambda} (\sigma(\xi^\varepsilon_x(u)) - \sigma(S^u x)) dw(u) \right|. \tag{4.26}$$

Using (3.13) together with (3.9) and (4.21) we conclude that

$$J^\varepsilon(x,t) \leqq \varepsilon K_{13} C_\delta^{(1)} \left(\int_0^t e^{2(\operatorname{Re}\lambda_1+\delta)(t-u)} E\chi_{u \leqq \tau_x^\varepsilon} |\xi_x^\varepsilon(u) - S^u x|^2 du \right)^{1/2}$$

(4.27)
$$\leqq \varepsilon^{2-\gamma} K_{13} C_\delta^{(1)} (C_{\delta,2\gamma}^{(11)} t)^{1/2} e^{(\operatorname{Re}\lambda_1+\delta)t},$$

for some $K_{13} > 0$. Applying Gronwall's inequality (see [3]) to the function

$$a_3(u) = e^{-(\operatorname{Re}\lambda_1+\delta)u} E |\xi_x^\varepsilon(u) - \eta_{0,x}^{\varepsilon,x}(u)| \chi_{u \leqq \tau_x^\varepsilon}$$

taken in (4.25) we get from (4.27) the assertion (4.22).

Now we can proceed to the proof of the theorems.

Proof of Theorem 2.1. Define

$$d_2(x) = \inf_{0 \leqq t < \infty} \operatorname{dist}(S^t x, \partial G). \tag{4.28}$$

Here we assume $x \in (A_1 \cup \mathcal{O}) \setminus \partial G$ and so $d_2(x) > 0$. Then by (4.21), (4.28) and Chebyshev's inequality we obtain

$$P\{\tau_x^\varepsilon \leqq t_{\gamma_0}(\varepsilon)\} \leqq P\left\{ \sup_{0 \leqq u \leqq \min(t_{\gamma_0}(\varepsilon), \tau_x^\varepsilon)} |\xi_x^\varepsilon(u) - S^u x| \geqq d_2(x) \right\}$$

(4.30)
$$\leqq C_{\delta,\gamma_4}^{(11)} \varepsilon^{\gamma_4} d_2^{-2}(x),$$

where $t_{\gamma_0}(\varepsilon)$ is given by (3.34) and

$$\gamma_4 = (\gamma_0 - \delta)(\operatorname{Re}\lambda_1 + \gamma_0)^{-1}. \tag{4.31}$$

Recall that the hyperplane $\Gamma_{\max}$ is tangent at $\mathcal{O}$ to the manifold $W_{\max}$ defined in §2. Therefore one can find $K_{14} > 0$ so that

$$\operatorname{dist}(z, W_{\max}) \leqq K_{14} r^2 \quad \text{for any } z \in \Gamma_{\max} \text{ and } r > 0, \quad \text{provided } |z| \leqq r. \tag{4.32}$$

Now using (4.22), (4.30)–(4.32) and Chebyshev's inequality we easily obtain from (3.35)–(3.37) that there are $\gamma_5, \gamma_6 > 0$ such that for any $\delta > 0$ and $x \in (A_1 \cup \mathcal{O}) \setminus \partial G$ one can choose $C_\delta^{(13)}(x) > 0$ so that

$$P\{\xi_x^\varepsilon(t_{\gamma_0}(\varepsilon)) \in W^\varepsilon(\delta, \gamma_0, 2\gamma_5)\} \leqq 1 - C_\delta^{(13)}(x) \varepsilon^{\gamma_6 \delta}, \tag{4.33}$$

where $W^\varepsilon(\delta, \gamma_0, \gamma_5) = \{z : \varepsilon^\delta r_{\gamma_0}(\varepsilon) \leqq |z| \leqq \varepsilon^{-\delta} r_{\gamma_0}(\varepsilon),\ \operatorname{dist}(z, W_{\max}) \leqq \varepsilon^{\gamma_5} r_{\gamma_0}(\varepsilon)\}$.

From chapter 9 of [3] it follows that for any $z \in W_{\max}$

$$\lim_{t \to \infty} t^{-1} \ln |S^{-t} z| = -\operatorname{Re}\lambda_1$$

and so for any $\delta>0$ there is $C_\delta^{(14)}>0$ such that

$$(C_\delta^{(14)})^{-1}|z|e^{(\mathrm{Re}\,\lambda_1-\delta)t}\leqq|S^tz|\leqq C_\delta^{(14)}|z|e^{(\mathrm{Re}\,\lambda_1+\delta)t}, \tag{4.34}$$

provided $z\in W_{\max}$ and $S^uz\in\tilde G$ for all $u\in[0,t]$, where $\tilde G\supset G\cup\partial G$ is the domain chosen in Lemma 4.2.

Set

$$\tilde t(z)=\inf\{t\geqq 0: S^tz\not\in\tilde G\};$$

then it is easy to see from Lemma 4.2 and (4.34) that

$$\sup_{z\in W^\varepsilon(2\delta,\gamma_0,\gamma_5)}\tilde t(z)\leqq t_{-4\delta\,\mathrm{Re}\,\lambda_1}(\varepsilon)-t_{\gamma_0}(\varepsilon), \tag{4.35}$$

provided ε is small enough.

Put also

$$\tilde{\tilde t}(z)=\inf\{t\geqq 0:|S^tz|\geqq\tfrac12\mathrm{dist}(\mathcal O,\partial G)\};$$

then by Lemma 4.2 and (4.34) one can see that

$$\inf_{z\in W^\varepsilon(2\delta,\gamma_0,\gamma_5)}\tilde{\tilde t}(z)\geqq t_{4\delta\,\mathrm{Re}\,\lambda_1}(\varepsilon)-t_{\gamma_0}(\varepsilon). \tag{4.36}$$

Finally, (4.9), (4.33), (4.35) and (4.36) give for $x\in(A_1\cup\mathcal O)\setminus\partial G$ that

$$P\{t_{4\delta\,\mathrm{Re}\,\lambda_1}(\varepsilon)\leqq\tau_x^\varepsilon\leqq t_{-4\delta\,\mathrm{Re}\,\lambda_1}(\varepsilon)\}\geqq 1-2C_\delta^{(13)}(x)\varepsilon^{\gamma_6\delta}, \tag{4.37}$$

provided ε is small enough. This together with the definition $t_\gamma(\varepsilon)$ in (3.34) yield (2.4) taking $\varepsilon\to 0$. The assertion (2.5) follows immediately from (4.9) and (4.10).

Proof of Theorem 2.2. From (3.34), (4.10) and (4.37) we easily get

$$\begin{aligned}
\big||\ln\varepsilon|^{-1}E\tau_x^\varepsilon-(\mathrm{Re}\,\lambda_1)^{-1}\big| &= \left||\ln\varepsilon|^{-1}\int_0^\infty P\{\tau_x^\varepsilon\geqq t\}dt-(\mathrm{Re}\,\lambda_1)^{-1}\right| \\
&\leqq\left||\ln\varepsilon|^{-1}\int_0^{t_{4\delta\mathrm{Re}\lambda}(\varepsilon)} P\{\tau_x^\varepsilon\geqq t\}dt-(\mathrm{Re}\,\lambda_1)^{-1}\right| \\
&\quad+|\ln\varepsilon|^{-1}\int_{t_{4\delta\mathrm{Re}\lambda}(\varepsilon)}^{t_{-4\delta\mathrm{Re}\lambda}(\varepsilon)} P\{\tau_x^\varepsilon\geqq t\}dt \\
&\quad+|\ln\varepsilon|^{-1}\int_{t_{-4\delta\mathrm{Re}\lambda}(\varepsilon)}^{\varepsilon^{-\frac12\gamma_6\delta}} P\{\tau_x^\varepsilon\geqq t\}dt \qquad (4.38)\\
&\quad+|\ln\varepsilon|^{-1}\int_{\varepsilon^{-\frac12\gamma_6\delta}}^{\infty} P\{\tau_x^\varepsilon\geqq t\}dt
\end{aligned}$$

$$\leqq 4\delta(1+4\delta)^{-1}(\operatorname{Re}\lambda_1)^{-1}+2C_\delta^{(13)}(x)\varepsilon^{\gamma_6\delta}(\operatorname{Re}\lambda_1)^{-1}(1+4\delta)^{-1}$$
$$+8\delta(\operatorname{Re}\lambda_1)^{-1}(1-16\delta^2)^{-1}+2C_\delta^{(13)}(x)\varepsilon^{\frac{1}{2}\gamma_6\delta}\,|\ln\varepsilon|^{-1}$$
$$+K_8(\tfrac{1}{2}\gamma_6\delta)\alpha_4^{-1}|\ln\varepsilon|^{-1}\exp(-\alpha_4\varepsilon^{-\frac{1}{2}\gamma_6\delta}),$$

provided $\varepsilon \leqq \varepsilon_1(\frac{1}{2}\gamma_6\delta)$.

Letting $\varepsilon \to 0$ in (4.38) and taking into account that $\delta > 0$ is arbitrarily small we shall get (2.6). The assertion (2.7) follows immediately from (4.9) and (4.10).

Proof of Theorem 2.3. By (4.35) and Lemma 4.2 we obtain that there are $K_{15}, \gamma_7 > 0$ such that

$$\sup_{z\in W^\varepsilon(2\delta,\gamma_0,\gamma_5)}\ \sup_{0\le t\le \hat{t}(z)} \operatorname{dist}(S^t z, W_{\max}) \leqq K_{15}\varepsilon^{\gamma_7}. \tag{4.39}$$

Therefore by (4.9), (4.33) and (4.39) we prove item (i) of Theorem 2.3. Item (iii) follows easily from (4.9) and (4.10).

To prove item (ii) define

$$\Gamma^\varepsilon(\delta,\gamma_0,\gamma_1)=\{z : \varepsilon^\delta r_{\gamma_0}(\varepsilon)\leqq |z| \leqq \varepsilon^{-\delta}r_{\gamma_0}(\varepsilon), \operatorname{dist}(z,\Gamma_{\max})\leqq \varepsilon^{\gamma_1}r_{\gamma_0}(\varepsilon)\},$$

where γ_0 and γ_1 are found in Lemma 3.4.

When $\nu = 1$, δ and ε are small enough then $\Gamma^\varepsilon(\delta,\gamma_0,\gamma_1)$ consists of two disjoint connected components $\Gamma_+^\varepsilon(\delta,\gamma_0,\gamma_1)$ and $\Gamma_-^\varepsilon(\delta,\gamma_0,\gamma_1)$ which are symmetrical with respect to $\mathcal{O}$. Notice that the process $\zeta_{0,\mathcal{O}}^{\varepsilon,x}(t)$ defined by (3.6) is also symmetrical with respect to $\mathcal{O}$, i.e., the probability distributions of $\zeta_{0,\mathcal{O}}^{\varepsilon,x}(t)$ and $-\zeta_{0,\mathcal{O}}^{\varepsilon,x}(t)$ coincide. Thus

$$P\{\zeta_{0,\mathcal{O}}^{\varepsilon,x}(t_{\gamma_0}(\varepsilon))\in\Gamma_-^\varepsilon(2\delta,\gamma_0,\tfrac{1}{2}\gamma_1)\} = P\{\zeta_{0,\mathcal{O}}^{\varepsilon,x}(t_{\gamma_0}(\varepsilon))\in\Gamma_+^\varepsilon(2\delta,\gamma_0,\tfrac{1}{2}\gamma_1)\}. \tag{4.40}$$

If δ and ε are small enough then also

$$\Gamma_-^\varepsilon(2\delta,\gamma_0,\tfrac{1}{2}\gamma_1)\cap\Gamma_+^\varepsilon(2\delta,\gamma_0,\tfrac{1}{2}\gamma_1)=\varnothing$$

and so by (4.40)

$$P\{\zeta_{0,\mathcal{O}}^{\varepsilon,x}(t_{\gamma_0}(\varepsilon))\in\Gamma_\pm^\varepsilon(2\delta,\gamma_0,\tfrac{1}{2}\gamma_1)\}\leqq\tfrac{1}{2}. \tag{4.41}$$

Since for $x \in (A_1 \cup \mathcal{O})\setminus\partial G$ it follows from (3.3) and (3.10) that

$$|\eta_{0,x}^{\varepsilon,x}(t_{\gamma_0}(\varepsilon)) - \zeta_{0,\mathcal{O}}^{\varepsilon,x}(t_{\gamma_0}(\varepsilon))| \leqq C^{(2)}e^{-\alpha_0 t_{\gamma_0}(\varepsilon)}$$

then by the choice of γ_0 in Lemma 3.4 we get for small ε that

$$P\{\zeta_{0,\mathcal{O}}^{\varepsilon,x}(t_{\gamma_0}(\varepsilon))\in\Gamma_\pm^\varepsilon(2\delta,\gamma_0,\tfrac{1}{2}\gamma_1)\}\geqq P\{\eta_{0,x}^{\varepsilon,x}(t_{\gamma_0}(\varepsilon))\in\Gamma_\pm^\varepsilon(\delta,\gamma_0,\gamma_1)\}. \tag{4.42}$$

On the other hand, by (3.35)–(3.37),

$$P\{\eta_{0,x}^{\varepsilon,x}(t_{\gamma_0}(\varepsilon))\in\Gamma_+^{\varepsilon}(\delta,\gamma_0,\gamma_1)\cup\Gamma_-^{\varepsilon}(\delta,\gamma_0,\gamma_1)\}\geqq 1-C_\delta^{(9)}\varepsilon^{\gamma_2\delta}, \tag{4.43}$$

provided δ and ε are small enough. Now by (4.41)–(4.43),

$$\tfrac{1}{2}\geqq P\{\eta_{0,x}^{\varepsilon,x}(t_{\gamma_0}(\varepsilon))\in\Gamma_\pm^{\varepsilon}(\delta,\gamma_0,\gamma_1)\}\geqq\tfrac{1}{2}-C_\delta^{(9)}\varepsilon^{\gamma_2\delta}. \tag{4.44}$$

Next, employing (4.22), (4.30)–(4.32) and Chebyshev's inequality we deduce from (4.44) that

$$\begin{aligned}\tfrac{1}{2}+C_\delta^{(15)}(x)\varepsilon^{\gamma_8\delta}&\geqq P\{\xi_x^{\varepsilon}(t_{\gamma_0}(\varepsilon))\in W_\pm^{\varepsilon}(\delta,\gamma_0,2\gamma_7)\}\\&\geqq\tfrac{1}{2}-C_\delta^{(15)}(x)\varepsilon^{\gamma_8\delta}\end{aligned} \tag{4.45}$$

(cf. (4.33)) for some $\gamma_7,\gamma_8>0$ and $C_\delta^{(15)}(x)>0$, where $W_+^{\varepsilon}(\delta,\gamma_0,2\gamma_7)$ and $W_-^{\varepsilon}(\delta,\gamma_0,2\gamma_7)$ are two disjoint connected components of $W^{\varepsilon}(\delta,\gamma_0,2\gamma_7)$.

Finally, using (4.9), (4.35) and (4.39) together with (4.45) we obtain assertion (ii) of Theorem 2.3 by letting $\varepsilon\to 0$, that completes the proof.

Appendix: Proof of (3.36) for all $x\in(A_1\cup\mathcal{O})\setminus\partial G$

Let DS_z^t be the differential of the dynamical system S^t at $z\in R^n$ and define the metric form

$$\sum_{i,j\leqq n}a_{ij}(z)dz^idz^j,\qquad z\in R^n, \tag{A1}$$

where $(a_{ij}(z))=(a^{ij}(z))^{-1}$ and the matrix $(a^{ij}(z))$ is given in (1.1). Denote by T_z the tangent space at $z\in R^n$ and for any $\xi,\eta\in T_z$ let $\langle\xi,\eta\rangle_z$ be the inner product of ξ and η in T_z generated by the metric form (A1).

The differential DS_z^t acts from T_z to T_{S^tz} and the adjoint operator $(DS_z^t)^*$ acts from T_{S^tz} to T_z and satisfies the property

$$\langle(DS_z^t)^*\eta,\xi\rangle_z=\langle\eta,DS_z^t\xi\rangle_{S^tz} \tag{A2}$$

for any $z\in R^n$, $\xi\in T_z$ and $\eta\in T_{S^tz}$.

Define

$$\begin{aligned}r_x^{\varepsilon}(t,\xi,\eta)&=(2\pi\varepsilon^2)^{-n/2}\left(\det{}_{S^tx}\int_0^t DS^{\tau}_{S^{t-\tau}x}(DS^{\tau}_{S^{t-\tau}x})^*d\tau\right)^{-1/2}\\&\times\exp\left\{-\frac{1}{2\varepsilon^2}\left\langle\left(\int_0^t DS^{\tau}_{S^{t-\tau}x}(DS^{\tau}_{S^{t-\tau}x})^*d\tau\right)^{-1}(\eta-DS_x^t\xi),(\eta-DS_x^t\xi)\right\rangle_{S^tx}\right\},\end{aligned} \tag{A3}$$

where $\xi\in T_x$, $\eta\in T_{S^tx}$ and the operator $\int_0^t DS^{\tau}_{S^{t-\tau}x}(DS^{\tau}_{S^{t-\tau}x})^*d\tau$ transforms the tangent space T_{S^tx} onto itself and so the determinant $\det_{S^tx}$ with respect to the inner product $\langle\ ,\ \rangle_{S^tx}$ is defined in a correct way.

Let $p_x^\varepsilon(s,z,t,y)$ be the transition density, with respect to the Euclidean volume in R^n, of the process $\xi_{s,z}^{\varepsilon,x}(t)$ defined by (3.3). One can easily see (cf. the formulas (6.6) in [6] and (4.3) in [7]) that

$$p_x^\varepsilon(0,0,t,y) = r_x^\varepsilon(t,0,y)(\det a_{ij}(S^tx))^{1/2}, \tag{A4}$$

where y is considered as a vector of R^n, as well as a vector of T_{S^tx}.

Let now $x \in (A_1 \cup \mathcal{O}) \setminus \partial G$. One has

$$\begin{aligned} P\{|\zeta_{0,\mathcal{O}}^{\varepsilon,x}(t)| \leqq \rho_1\} &= \int_{\{y:|y|\leqq\rho_1\}} p_x^\varepsilon(0,0,t,y)dy \\ &\leqq \int_{\{\eta:\|\eta\|_{S^tx}\leqq\rho_2\}} r_x^\varepsilon(t,0,\eta)d_{S^tx}\eta, \end{aligned} \tag{A5}$$

where $\rho_2 = K_{16}\rho_1$, $K_{16} > 0$ is a constant independent of ρ_1, x, t; $d_{S^tx}\eta$ is the element of the Euclidean volume generated by the inner product $\langle\ ,\ \rangle_{S^tx}$ in T_{S^tx} and $\|\eta\|_{S^tx}$ is the norm of η with respect to the same inner product.

Let Γ^u and Γ^s be the eigenspaces of the matrix Λ from (2.1) corresponding to the eigenvalues of Λ with positive and negative real parts, respectively. Recall that $\Gamma_{\max}$ denotes the eigenspace of Λ corresponding to $\lambda_1, \cdots, \lambda_\nu$ in (2.3).

Choose $t_0 > 0$ big enough so that $S^{t_0}x$ is close enough to $\mathcal{O}$ and set, for $t \geqq 0$,

$$\tilde{\Gamma}^u_{S^tx} = DS^{t-t_0}_{S^{t_0}x}\Gamma^u \quad \text{and} \quad \Gamma^{\max}_{S^tx} = DS^{t-t_0}_{S^{t_0}x}\Gamma_{\max}. \tag{A6}$$

It is easy to see that if t_0 is big enough then

$$\tilde{\Gamma}^u_{S^tx} \to \Gamma^u \quad \text{and} \quad \Gamma^{\max}_{S^tx} \to \Gamma_{\max} \qquad \text{as } t \to \infty, \tag{A7}$$

and also

$$\|DS^{-t}_{S^{t+\tau}x}\eta\|_{S^\tau x} \leqq K_{17}e^{-\gamma_9 t}\|\eta\|_{S^{t+\tau}x} \tag{A8}$$

for any $\eta \in \tilde{\Gamma}^u_{S^{t+\tau}x}$ and all $t, \tau \geqq 0$ with $\gamma_9, K_{17} > 0$ independent of t, τ and η.

Next, for some subsequence $t_i \to \infty$ the subspaces $DS^{-t_i}_{S^{t_i}x}\Gamma^s$ converge to a subspace $\tilde{\Gamma}$. Set

$$\tilde{\Gamma}^s_{S^tx} = DS^t_x\tilde{\Gamma} \qquad \text{for } t \geqq 0. \tag{A9}$$

One can verify that $\tilde{\Gamma}^s_{S^tx} \to \Gamma^s$ as $t \to \infty$ and

$$\|DS^t_{S^\tau x}\xi\|_{S^{t+\tau}x} \leqq K_{18}e^{-\gamma_{10}t}\|\xi\|_{S^\tau x} \tag{A10}$$

for any $\xi \in \tilde{\Gamma}^s_{S^\tau x}$ and all $t, \tau \geqq 0$, where $\gamma_{10}, K_{18} > 0$ are independent of t, τ and ξ.

In fact, under assumptions of §2 it follows from [9] that the tangent bundle T_G restricted to the domain G has an invariant under DS^t continuous splitting $T_G = \tilde{\Gamma}^u \oplus \tilde{\Gamma}^s$, where $\tilde{\Gamma}^u$ and $\tilde{\Gamma}^s$ are expanding and contracting subbundles, respectively. However we shall need here this splitting just for one trajectory $\{S^t x, t \geqq 0\}$ and so the spaces $\tilde{\Gamma}^u_{S^t x}, \tilde{\Gamma}^s_{S^t x}$ and $\tilde{\Gamma}^{\max}_{S^t x}$, constructed above, will be enough for our purposes.

Define the family of operators

$$\widetilde{DS}^t_{S^\tau x} : T_{S^\tau x} \to T_{S^{t+\tau}x}, \qquad t, \tau \geqq 0$$

acting as follows:

$$\text{(A11)} \qquad \widetilde{DS}^t_{S^\tau x}\xi = DS^t_{S^\tau x}\xi \qquad \text{if } \xi \in \tilde{\Gamma}^u_{S^\tau x}$$

and

$$\text{(A12)} \qquad \widetilde{DS}^t_{S^\tau x}\eta = \frac{\|\eta\|_{S^\tau x}}{\|DS^t_{S^\tau x}\eta\|_{S^{t+\tau}x}} DS^t_{S^\tau x}\eta \qquad \text{if } \eta \in \tilde{\Gamma}^s_{S^\tau x}.$$

In the same way as in lemma 4.2 from [7] we obtain that

$$\text{(A13)} \qquad K_{19}^{-1} \leqq \left(\det{}_{S^t x}\left(\int_0^t DS^\tau_{S^{t-\tau}x}(DS^\tau_{S^{t-\tau}x})^* d\tau\right)\right)^{-1/2} |\mathrm{Jac}\, \widetilde{DS}^t_x| \leqq K_{19}$$

for some $K_{19} > 0$ independent of t, where $\mathrm{Jac}\, \widetilde{DS}^t_x$ is the Jacobian of the linear map $\widetilde{DS}^t_x : T_x \to T_{S^t x}$ with respect to inner products $\langle \ , \ \rangle_x$ and $\langle \ , \ \rangle_{S^t x}$.

Next, for any $\xi \in T_x$

$$\text{(A14)} \qquad \left\langle \left(\int_0^t DS^\tau_{S^{t-\tau}x}(DS^\tau_{S^{t-\tau}x})^* d\tau\right)^{-1} \widetilde{DS}^t_x \xi, \widetilde{DS}^t_x \xi \right\rangle_{S^t x} \geqq K_{20}\langle \xi, \xi \rangle_x,$$

for some $K_{20} > 0$ independent of ξ and t. Indeed, it suffices to prove that the selfadjoint operator

$$W_1(t) = \left(\widetilde{DS}^{-t}_{S^t x} \int_0^t DS^\tau_{S^{t-\tau}x}(DS^\tau_{S^{t-\tau}x})^* d\tau (\widetilde{DS}^{-t}_{S^t x})^*\right)^{-1}$$

acting from T_x to T_x has eigenvalues which are greater than or equal to K_{20} or, equivalently, that the eigenvalues of the operator $W_1^{-1}(t)$ are less than or equal to K_{20}^{-1}. But the last statement can be proved in the same way as inequality (4.34) of [7].

For every $\xi \in T_x$ there is the unique decomposition $\xi = \xi^{\max} + \xi^\perp$, where $\xi^{\max} \in \Gamma^{\max}_x$ and $\xi^\perp$ is orthogonal to $\Gamma^{\max}_x$ with respect to the inner product $\langle \ , \ \rangle_x$.

Set $U(t, \rho_2) = \{\eta \in T_{S^t x} : \|\eta\|_{S^t x} \leqq \rho_2\}$. Employing (A3), (A6), (A13), (A14) and Fubini's theorem one gets

$$
\begin{aligned}
&\int_{U(t,\rho_2)} r_x^\varepsilon(t,0,\eta)\,d_{S^t x}\eta \\
&\leqq K_{19}(2\pi\varepsilon^2)^{-n/2}\int_{\widetilde{DS}{}^{-t}_{S^t x}U(t,\rho_2)} \exp\left(-\frac{K_{20}}{2\varepsilon^2}\langle\xi,\xi\rangle\right)d_x\xi \\
&\leqq K_{21}\varepsilon^{-\nu}\int_{\Gamma_x^{\max}\cap\widetilde{DS}{}^{-t}_{S^t x}U(t,\rho_2)} \exp\left(-\frac{K_{20}}{2\varepsilon^2}\langle\xi^{\max},\xi^{\max}\rangle\right)d_x\xi^{\max} \qquad \text{(A15)}\\
&\leqq K_{21}\varepsilon^{-\nu}\,\mathrm{vol}_x^{\max}(\Gamma_x^{\max}\cap DS^{-t}_{S^t x}U(t,\rho_2)) \\
&\leqq K_{22}\varepsilon^{-\nu}\rho_2^{\nu}\left|\mathrm{Jac}(\widetilde{DS}{}^{-t}_{S^t x})_{\Gamma^{\max}_{S^t x}}\right| \\
&= K_{22}\varepsilon^{-\nu}\rho_2^{\nu}\left|\mathrm{Jac}(\widetilde{DS}{}^{t}_{x})_{\Gamma_x^{\max}}\right|^{-1},
\end{aligned}
$$

for some $K_{21}, K_{22}>0$ independent of t and ε, where $\mathrm{vol}_x^{\max}$ is the volume on the subspace $\Gamma_x^{\max}$ with respect to the inner product $\langle\ ,\ \rangle_x$, $\nu=\dim\Gamma_{\max}=\dim\Gamma_x^{\max}$ and we denote by $(H)_Y$ the restriction of the linear map $H: T_y\to T_z$ on the linear subspace Y of T_y and by $\mathrm{Jac}(H)_Y$ the Jacobian of the linear map $(H)_Y: Y\to Z=HY$ with respect to inner products $\langle\ ,\ \rangle_y$ and $\langle\ ,\ \rangle_z$.

Since $|\mathrm{Jac}(e^{t\Lambda})_{\Gamma_{\max}}|=|\det(e^{t\Lambda})_{\Gamma_{\max}}|$ then by the definition of $\Gamma_{\max}$ it follows that

$$\lim_{t\to\infty} t^{-1}\ln|\mathrm{Jac}(e^{t\Lambda})_{\Gamma_{\max}}|=\nu\,\mathrm{Re}\,\lambda_1. \tag{A16}$$

Taking into account that $DS_{\mathcal{O}}^t=e^{t\Lambda}$, DS_z^t depends smoothly on z,

$$DS_x^t=\widetilde{DS}{}_x^t \quad \text{on } \Gamma_x^{\max},\quad S^t x\to\mathcal{O}$$

as $t\to\infty$ and using (A6), (A7) and (A16) we obtain that

$$\lim_{t\to\infty} t^{-1}\ln|\mathrm{Jac}(\widetilde{DS}{}_x^t)_{\Gamma_x^{\max}}|=\nu\,\mathrm{Re}\,\lambda_1. \tag{A17}$$

One can see from here that for each $\kappa>0$ there is $C_\kappa^{(16)}>0$ independent of $t\geqq 0$ such that

$$|\mathrm{Jac}(DS_x^t)_{\Gamma_x^{\max}}|\geqq C_\kappa^{(16)}e^{\nu(\mathrm{Re}\,\lambda_1-\kappa)}. \tag{A18}$$

Using (A15) with $\rho_2=K_{16}\rho_1=K_{16}(\varepsilon^\delta r_{\gamma_0}(\varepsilon)+C^{(2)}e^{-\alpha_0 t})$ we get by (3.3), (3.10), (A5), (A15) and (A18) that

$$
\begin{aligned}
&P\{|\eta_{0,x}^{\varepsilon,x}(t)|\leqq\varepsilon^\delta r_{\gamma_0}(\varepsilon)\}\leqq P\{|\zeta_{0,\mathcal{O}}^{\varepsilon,x}(t)|\leqq\varepsilon^\delta r_{\gamma_0}(\varepsilon)+C^{(2)}e^{-\alpha_0 t}\} \qquad \text{(A19)}\\
&\qquad\leqq K_{16}^{\nu}K_{22}(C_\kappa^{(16)})^{-1}\varepsilon^{-\nu}e^{-(\mathrm{Re}\,\lambda_1-\kappa)\nu t}(\varepsilon^\delta r_{\gamma_0}(\varepsilon)+C^{(2)}e^{-\alpha_0 t})^{\nu}.
\end{aligned}
$$

Taking $t=t_{\gamma_0}(\varepsilon)$, $\gamma_0=\frac{1}{4}\min(\alpha_0,1)$, $\kappa=\frac{1}{4}\delta(\mathrm{Re}\,\lambda_1+\gamma_0)$ and δ small enough we shall get (3.36) from (3.34) and (A19).

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